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Quantifying the cost of excess market thickness in timber sale auctions☆

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ABSTRACT

In auctions with endogenous entry, theory predicts that too many potential bidders, or the excess market thickness, may actually decrease the seller's expected revenue and the social welfare generated by the auction. This paper proposes a computationally easy method for estimating the optimal number of potential bidders in timber sale auctions with endogenous entry and an uncertain number of active bidders and then quantifies the cost of excess market thickness. It is found that the welfare loss due to the excess market thickness is moderate in this market.

1. Introduction

This paper uses the structural approach to analyze and quantify the cost, both to the seller and the society, due to excess market thickness in timber sale auctions. Every year, both the federal and state governments in the United States sell many timber harvesting rights to logging companies using auctions. The total revenue from such auctions is large. For example, in the state of Michigan alone, where the data used in the empirical part of this paper come from, every year, $20 to $30 million worth of timber are sold through auctions. Because of the large amount of money involved, a central research question, which is of interest to both academic researchers as well as government policy makers, is which auction mechanism to use such that the seller's revenue and/or the welfare of the society is maximized.

From an auction mechanism design point of view, an auction is characterized by three instruments that are under the seller's control. These instruments determine the likely outcomes of the auctions. First, the seller decides which auction format to use.[1] Second, the seller sets the reserve price and decides whether to adopt a public reserve price strategy or a secret reserve price strategy. Finally, the seller decides on how many bidders to invite for the auction, that is, the number of potential bidders.

The theoretical relationships between the three auction instruments and the likely outcomes of the auction are well understood in the literature. First, in the standard auction model where bidders' private values are independent (IPV framework), it is well known that the four auction formats mentioned above yield the same expected revenue for the seller (Vickrey, 1961; Riley and Samuelson, 1981; Myerson, 1981).[2] Regarding the reserve price, in the IPV framework, both Laffont and Maskin (1980) and Riley and Samuelson (1981) have shown that the optimal reserve price is above the seller's own value of the auctioned object and depends on the density of bidders' private value distribution. Recently, Levin and Smith (1994) find that in auctions with endogenous entry, the optimal reserve price equals the seller's own value of the auctioned good. Finally, for the relationship between the number of potential bidders and the outcome of the auction, in the IPV framework, the seller's expected revenue is a monotone increasing function of the number of potential bidders and therefore, the optimal strategy for the seller is to invite as many potential bidders as possible. However, when the auction environment changes to the common value (CV) framework or to auctions with endogenous entry, this result no longer holds. For example, Levin and Smith (1994) find that there is a non-monotone relationship between

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[1] For single-unit auctions, four auction formats are often used. They are the first-price sealed-bid auction, second-price sealed-bid auction, English auction and Dutch auction. In multi-unit auctions, within each of the four auction formats listed above, the seller can further choose between a discriminatory auction and a uniform-price auction.

[2] When bidders' private values are interdependent or affiliated, Milgrom and Weber (1982) show that the English auction yields the highest expected revenue to the seller and the first-price sealed bid auction yields the lowest expected revenue to the seller.

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the number of potential bidders and the seller's revenue or the social welfare for auctions with entry and a certain number of active bidders. Therefore, inviting as many potential bidders as possible is not an optimal strategy. Li and Zheng (2008) extend their result to auctions with entry and an uncertain number of active bidders.

On the other hand, since many auction data are available from federal and state government agencies, it provides a unique opportunity for empirical researchers to evaluate the usefulness of theoretical models using field data. Indeed, starting with Paarsch (1992), various econometric methods have been proposed to estimate and test the implications of various auction models. Examples include, Guerre et al. (2000), Li et al. (2000), Haile et al. (2002) and Athey and Haile (2002), to mention just a few. In this literature, some studies focus upon the welfare implications of the first two instruments of auction mechanism design, that is, auction format and reserve price. Regarding the auction format, for example, Athey et al. (2004) find that sealed bid auctions attract more small bidders, shift the allocation towards these bidders, and can also generate higher revenue than open ascending auctions in U.S. Forest Service timber auctions. For another example, Kang and Puller (2007) find that discriminatory auctions yield a larger expected revenue as well as better efficiency than uniform-price auctions in the multi-unit Korean treasury auctions. Regarding the optimal use of reserve price, both Paarsch (1997) and Li and Perrigne (2003) find that switching to the optimal reserve price strategy, the seller's revenue can increase substantially in timber sale auctions.

However, to date, there exists no empirical study examining the welfare implications of the optimal use of the third instrument, that is, number of potential bidders. As discussed above, in the standard IPV auction, this is not an issue as the seller's revenue is monotone increasing in the number of potential bidders. But for auctions with endogenous entry as for the timber sale auctions studied in this paper, it matters. According to the theory, limiting the number of potential bidders may benefit the seller, and/or the society. How much can the seller gain by adopting an optimal strategy regarding the number of potential bidders? Furthermore, in order to adopt such a strategy, policy makers need to know the optimal number of potential bidders in the first place. This paper aims to answer these questions and fill the gap between the theoretical and empirical auction studies with respect to the optimal use of the number of potential bidders and empirically quantify the cost of allowing a sub-optimal number of potential bidders in timber sale auctions.

In order to achieve these two goals, a computationally easy approach is proposed to estimate a structural auction model with endogenous entry and uncertain number of active bidders for timber sale auctions.[3] Using the estimated model primitives, the optimal number of potential bidders is calculated for each auction in the dataset. Then new auction outcomes under the optimal number of potential bidders are simulated. It is found that by switching to auction mechanisms with optimal number of potential bidders, the median winning bid increases \$45.46 and the median social welfare increases \$1006.29, which account for approximately 0.15% and 3.39% of the median winning bid observed in data, respectively. Therefore, the welfare loss due to excess market thickness is moderate in this market.

This paper makes several contributions to the literature. First, this is the first paper that quantifies the welfare implications of allowing a sub-optimal number of potential bidders into the auction. It demonstrates the empirical importance of the number of potential bidders as a tool for the seller in the process of designing optimal auction mechanism. As mentioned above, most of the previous empirical studies in this literature focus on other aspects of the design like auction format and reserve price. Second, theoretically, in an auction model with entry and uncertain number of active bidders, it is proved that from a seller's point of view, the optimal number of potential bidders is the maximum number of potential bidders that still induce all the potential bidders to enter into the auction and become active bidders. This generalizes the result in Levin and Smith (1994) to more general auction models with entry. Finally, this paper offers a computationally easy approach to estimate the optimal number of potential bidders in auctions with entry and uncertain number of active bidders, thus providing an easy way for policy makers to assess the optimality of their auction designs.

The rest of the paper is organized as follows. Section 2 describes the data and discusses results from a reduced-form analysis of the data. The auction model and theoretical results are presented in Section 3. Section 4 outlines the estimation method and reports the estimation results. Counterfactual analyses are conducted in Section 5 to quantify the cost of excess market thickness. Finally, Section 6 concludes. Technical proofs are collected in Appendix A.

2. Data

2.1. Timber auctions

The data used in this paper come from the timber sale auctions organized by Michigan Department of Natural Resources (MDoNR), the state agency in charge of the management of state forests in Michigan. The market mechanism chosen by MDoNR for its sales of standing timber is the standard first-price sealed-bid auctions with a public reserve price. MDoNR usually advertises the auctions 4 to 6 weeks prior to the sale date. In the advertisement, it provides detailed information on the timber, such as what species are in the lot, the volume of each species, the percentage of saw timbers and the minimum acceptable bid (the public reserve price). Also, the location of the lot is given and any bidder can inspect it before submitting a bid. Industry sources confirm that bidders usually cruise the lot before submitting a bid. This is because different lots of timber differ significantly in terms of timber quality and harvesting costs, resulting in different values for the bidder. Also, since timbers grow slowly, each lot will not be harvested again within 60 years in Michigan. Therefore, each lot is unique and experience with other lots may provide limited information for the current lot. Furthermore, the advertisement gives an estimation error range, measuring how precise the agency's estimate of the volume of timber. For example, if the error range for one auction is 20%, it means that the actual volume of the timber on the lot can differ from the volume given in the advertisement by 20%, both from above and from below. This measure varies significantly across auctions, ranging from 0% to 66.29%, which gives the bidders strong incentives for cruising the lot themselves to develop a better idea about the volume and hence the value of the lot.

The potential bidders for these timbers are private firms, usually local logging companies and sawmills. Interested firms submit their sealed bids before the bid opening time on the sale date. The sales manager collects and opens the bids, ranks them, and announces the bidder with the highest bid as the winner, provided his bid is above the public reserve price stated in the advertisement. The winner pays his bid.[4] The unsold lots may be proposed for sale in the future, upon the sales manager's decision. It is important to note that, because of its public nature, MDoNR has a strong commitment in its reserve price.

[3] Empirical entry models are first introduced into the industrial organization literature by Bresnahan and Reiss (1990), who study entry in monopoly/oligopoly markets with complete information. Recent extensions include Tamer (2003), Ciliberto and Tamer (2007) and Seim (2006). In these models, firms' post-entry profits are often modelled using reduced forms, while in auctions models with entry, bidders' post-entry profits are modelled using a structural auction model.

[4] Bidders propose a total bid for the lot and payments are equal to bids and are not based on actual value harvested. Therefore, there is no skewed bidding as in Athey and Levin (2001).

Table 1
Variable definitions and summary statistics.

Variable	Explanation	# of obs	Mean	S. D.	Min	Max
Bids	Bids	1209	40,824.48	36,568.68	671.91	229,985.70
Winbid	Winning bid	314	42,729.71	39,666.95	681.90	229,985.70
Reserve	Reserve price	332	28,205.10	27,805.67	601.90	195,283.10
Acre	Acres of the lot	332	72.14	56.55	4	297
Range	Volume estimation error range (%)	332	16.80	8.26	0	66.29
Years	Number of allowed harvest years	332	2.09	0.21	0.08	3.17
Distance	Miles of driving distance	4326	78.14	106.93	3.1	1461
Logging	1 if bidder is a logging company	87	0.71	0.46	0	1
Actual	Number of actual bids	332	3.64	2.30	0	11
Potential	Number of potential bidders	332	12.92	4.90	3	23

The sample consists of 332 auctions with a total of 1209 bids. 314 auctions receive at least 1 bid. The number of potential bidder-auction links is 4326.

This means that there is no possible second round with bargaining between the highest bidder and MDoNR, which is frequently observed in auctions organized by private cooperatives, as is the case in Elyakime et al. (1994, 1997). This is also confirmed by the data that no bidder submits a bid lower than the public reserve price.

MDoNR is a decentralized institution, and each region has its own office. The data analyzed in this paper come from the Baldwin office, which is located in the upper western part of lower peninsula Michigan. Each local office is in charge of the management of the state forestry on its territory. The sales manager of each local agency decides every year about the volume of timber to be sold through auctions and is responsible for organizing auctions. Sales are important for MoDNR. Each year MoDNR receives about \$20 to \$30 million from its timber sales.

2.2. Summary statistics and reduced form analysis

For each auctioned lot, the following variables are observed from the bidding summary sheet: the reserve price announced by MDoNR, the actual bids and bidders' identities, the area of the lot in acres, the volume estimation error range given by MDoNR and the number of allowed harvest years. We supplement these data with data on bidder type (logging firm or sawmill) and the driving distance between where the bidder is located and the auction site.[5] The data used in this paper include all auctions held by the Baldwin field office from January, 1999 to August 2004. This gives 332 auctions with a total of 1209 bids. These bids come from 87 bidders, of whom 62 are logging companies (with 1026 bids and winning 265 auctions) and 15 are sawmills (with 183 bids and winning 49 auctions). Most of the logging companies participating in these auctions employ between 1 and 10 people, while sawmills are significantly larger, usually having tens or hundreds of employees. For each auction, a rough measure for the number of potential bidders is constructed using the total number of bidders that submit an actual bid for any timber sale auction that is held by the same MDoNR regional office in the same month.

Table 1 provides summary statistics. First, it shows that timber lots differ in size, which can be seen from the statistics on the acre and the reserve price variables. Second, a great variability in bids is observed. Part of this variability is due to the heterogeneity of the lots and the rest is due to differences in bidders' valuations. Third, the entry

Table 2
Random effects regression of the log of bids on auction characteristics.

Variable	Coefficient	*t*-statistic
log(Reserve)	0.9772	89.82
Acre	−0.0004	−0.02
Range	−0.1164	−1.08
Years	0.0645	1.85
Logging	−0.0058	−0.36
Distance	0.0001	1.26
Constant	0.3608	2.95
Adjusted R^2	0.9528	

behavior is present in the dataset. The average number of potential bidders (using the rough measure) for an auction in the dataset is 12.92 but the average number of actual bidders is only about 3.64. That only 28.17% of the potential bidders actually submit their bids is a strong evidence of bidders' entry behavior. In order to bid for one lot, a bidder needs to incur some costs to cruise the lot himself and develop a value for the lot. The entry behavior indicates that bidders rationally take the entry cost into account when deciding whether to enter into the final bidding process. Fourth, there is also significant variability in the volume estimation error range given in the timber advertisement. The average estimation error range is 16.80%, with the minimum at 0% and the maximum at 66.29%. This variability may induce different entry behavior by the bidders since a higher estimation error range may indicate that the geographic characteristics of the lot is complex, leading to higher cruising costs for the potential bidders as well as higher transportation costs for the winning bidder. This may partly explain the fact that the number of actual bidders also varies significantly across auctions, even after controlling for auction heterogeneity. Fifth, the data reveal that on average MDoNR gives the winning bidder 2 years to harvest the timber. Industry sources confirm that the actual harvest time needed is usually less than 1 month. With the additional facts that MDoNR divides the timber lots into small pieces[6] and the winning probability is roughly the same for all the bidders, bidders in this market are not likely to be constrained by their harvest capacities. Sixth, the average driving distance between the location of a potential bidder and the auction site is about 78 miles, indicating local bidders are the main participants of these auctions. Finally, in the dataset, 18 auctions out of 332 receive no bids. This accounts for about 5.42% of the auctions. Previous studies of timber sale auctions usually focus on studying auctions with at least 2 bids as in Haile (2001) and Li and Perrigne (2003). Auction models with entry considered in this paper endogenize the bidders' entry behavior and can accommodate the case of no bids. Therefore, in the estimation part, data from auctions that receive no bids are also used, which may increase the efficiency of the estimates compared with other methods that exclude those auctions from the analysis.

To assess empirically the variability in bids as well as the effects of heterogeneity among auctioned lots, a random effects panel data regression is conducted by regressing the logarithm of bids on a complete vector of variables, following the literature in adopting the so-called reduced-form approach.[7] These variables are logarithm of the reserve price, the acres of the lot, the volume estimation error range, the number of the allowed harvest years, the type of the bidder and the driving distance. The results are presented in Table 2.

First, note that the R^2 statistic for this regression is 0.9528, indicating that the regression contains the most important exogenous variables. Second, Table 2 shows that the logarithm of the reserve price is highly significant. The reserve price seems to be the most accurate variable to capture different lot species, quality and location.

[5] The driving distance is calculated as follows. First, the longitude and the latitude information of both the center of the 5-digit zip code where the bidder is located and the center of the auction site are obtained. Then, the driving distance is calculated by Google Map using the longitude and latitude information.

[6] As can seen from Table 1, each lot is worth \$42729.71 on average.

[7] A random effects panel data regression is adopted here because multiple bids from the same auction are observed, making the data structure very similar to that of a panel data.

Third, the number of years of allowed harvest time for one lot has a small but significant (at the 10% level) positive effect on the bids. All other variables are not significant at the 10% significance level. In particular, the bidder type variable seems to have no effect on the bids. In addition, logging companies entered 84.86% (1026/1209) of the bids and won 84.39% (265/314) of the successfully sold auctions, implying that the winning probabilities are roughly the same for logging companies and sawmills. These evidence indicate that symmetry in terms of bidder type is a good assumption for this dataset. These findings are in contrast with those of Athey et al. (2004), who report that sawmills bid 24% higher than logging companies and have higher winning probability. These differences may come from the fact that data used by Athey et al. (2004) come from U.S. federal timber sale auctions. The average winning bid in their dataset is about $202,915 for open auctions and $120,475 for sealed bid auctions, which are significantly larger than that of the dataset used here. It might be the case that because of the small value of the auctions organized by MDoNR, there is more homogeneity across the firms participating in these auctions.

Comparing with other studies on timber sale auctions in the literature, the size of the auctions studied here is most close to that of Li and Perrigne (2003), who study timber auctions organized by the French Forest Service with an average value of 111,806 1993 French franc (about 27,359 2005 U.S. dollars). It is significantly less than those of Paarsch (1992, 1997), who study timber auctions in British Columbia in Canada with an average value of 94,200 1987 Canadian dollars (about 123,966 2005 U.S. dollars) and Haile (2001) who studies timber sale auctions by U.S. Forest Service with an average value of 632,232 1983 U.S. dollars (about 1,233,821 2005 U.S. dollars). In terms of the type of bidders, Haile (2001) reports that bidders in his dataset are predominantly sawmills, while other studies provide very little information on the type of the bidders. In our study, bidders are predominantly logging companies. These evidence indicate that when the value of auctions is large, sawmills are more likely to participate in these auctions, while small auctions attract mainly logging companies.

Also note that the distance variable has no significant effect on the bids as well. For auction models with entry, an important modelling issue is whether the entry costs should be assumed to be the same or different across bidders. One likely reason for entry cost to be different is that the distances between the bidder location and the location of auction site are different for different bidders. If bidders have different entry costs, they would bid differently, hoping to recover the entry cost. The evidence from Table 2 shows that this is not the case. To further examine this issue, a second reduced form regression is conducted by regressing the potential bidders' entry decisions on the covariates employing a random effects probit model. If bidders have different entry costs, they would have different probabilities of entering into the auction. The results are collected in Table 3. Again, the distance variable is insignificant. These results indicate that symmetry in terms of the entry cost might also be a good assumption for this dataset. The results from this regression can also be regarded as supporting evidence for the conjecture that the current choice of how to measure potential bidders is consistent with the assumption of symmetric entry cost.[8]

Different from most of the other studies on timber sales in the literature, an important feature of the model below is that it endogenizes bidders' entry decision, that is, the number of active bidders and hence the number of actual bidders is assumed to be endogenously determined rather than exogenously given.[9] In order to assess the validity of this assumption, the relationship between the number of actual bidders and other auction characteristics is further analyzed using the reduced-form approach. To allow for the discrete

Table 3
Random effects probit regression of potential bidders' entry decisions on auction characteristics.

Variable	Coefficient	*t*-statistic
log(Reserve)	0.3372	4.30
Acre	−0.3851	−2.85
Range	−1.5321	−2.05
Years	0.0938	0.36
Distance	−0.0000	−0.05
Constant	−3.9629	−4.44
Log likelihood	−2507.02	

values taken by the number of actual bidders, a Poisson regression model as well as a negative binomial regression model is considered. The results are given in Table 4.

First, Table 4 shows that the reserve price has a significant positive effect on the number of actual bidders. If the profit from a lot is proportional to the value of the lot, then a lot with larger size would induce more bidders to enter. Also, it is found that the acres and the volume estimation error range have negative effects on the number of actual bidders. This may be due to the fact that bidders take these variables as indicators of the higher difficulty and larger costs in cruising of the lot. Therefore, fewer potential bidders will enter those auctions.

Though very informative, these reduced-form analyses have little to say regarding the optimality of the auction mechanism adopted by MDoNR and by how much MDoNR could eventually increase its profit by switching to another mechanism like limiting the number of potential bidders and hence reducing the market thickness. It is the purpose of the next sections to respond to these important economic questions. On the other hand, though not a formal test, the empirical evidence from the reduced-form approach, together with the described auction mechanism lends support to the belief that a symmetric first-price sealed-bid auction model with endogenous entry is a first approximation for this timber sale auction market.

3. The model

The model considered here is similar to that of Li and Zheng (2008), with the only difference that a high bid auction model is considered here while Li and Zheng (2008) consider a low bid auction model. The government auctions a single and indivisible lot of timber. All bids are collected simultaneously. The lot is sold to the highest bidder who pays his bid to the government, provided the bid is at least as high as a reservation price p_0. There are N potential bidders in the market. Each potential bidder is risk-neutral with a private value of the lot v. Moreover, each potential bidder must incur an entry cost k such as lot evaluation, bid preparation and acquiring information to learn his private value of the lot. Thus, the auction is modelled as a two-stage game.

At the first stage, knowing the number of potential bidders N, each potential bidder learns the specifications of the lot and the entry cost, calculates his expected profit from entering the auction conditioning on his winning, and then decides whether or not to participate in the auction and actually submit a bid. Each potential bidder has the same entry cost k, which is a common knowledge to all potential bidders. Moreover, all potential bidders do not draw their private values until after they decide to enter the auction.

After the first entry stage, the n buyers who decide to enter the auction become active bidders and learn their own values of the lot. The value of the lot v to a buyer is drawn from a distribution $F(\cdot)$ with support $[\underline{v},\bar{v}]$. $F(\cdot)$ is twice continuously differentiable and has a density $f(\cdot)$ that is strictly positive on the support. In the independent private value paradigm, when forming his bid, each active bidder knows his private value v, but does not know other active bidders' private values. On the other hand, each active bidder knows that all private values are

[8] I thank an anonymous referee for raising this possibility.

[9] An exception is Athey et al. (2004), who compare open and sealed-bid auctions using data from U.S. Forest Service.

Table 4
Poisson and negative binomial regression of the number of actual bidders on auction characteristics.

Variable	Poisson		Negative binomial	
	Coefficient	*z*-statistic	Coefficient	*z*-statistic
log(Reserve)	0.2021	3.24	0.2050	2.87
Acre	−0.0028	−2.72	−0.0029	−2.47
Range	−0.0179	−2.45	−0.0196	−2.28
Years	−0.0978	−0.16	−0.1169	−0.17
Constant	0.6265	0.15	0.7614	0.17
Log likelihood	−314.3292		−312.2022	

independently drawn from $F(\cdot)$, which is a common knowledge to all bidders. Moreover, the reserve price p_0 is common knowledge with $p_0 \in [\underline{v}, \bar{v}]$ and an active bidder submits his bid (hence becomes an actual bidder) only when his private value v is greater than or equal to p_0.

As a result, all bidders are identical *a priori* and the game is symmetric.

3.1. First-stage: mixed strategy entry

Denote $E\pi(b, v|q^*, N)$ as the payoff for the actual bidder who optimally bids b using a Bayesian–Nash equilibrium strategy given his own value v, the unique equilibrium entry probability q^*, the number of potential bidders N and the belief that he is the winner. Therefore, the ex ante expected payoff for a potential bidder without knowing v is

$$\int_{p_0}^{\bar{v}} E\pi(b, v|q^*, N) f(v) \mathrm{d}v.$$

From a game-theoretic viewpoint, a rational potential bidder will participate in the auction only when the ex ante expected payoff exceeds his corresponding entry cost, that is, if $\int_{p_0}^{\bar{v}} E\pi(b,v|q^*,N)\ f(v)\mathrm{d}v \geq k$. Otherwise, remaining outside of the auction is an optimal strategy. The mixed strategy equilibrium at this stage is a unique q^* such that the ex ante equilibrium expected payoff equals the entry cost, that is,

$$\int_{p_0}^{\bar{v}} E\pi(b, v|q^*, N) f(v) \mathrm{d}v = k. \tag{1}$$

Since the potential bidders are ex ante symmetric in this model, the probability that there will be j active bidders $(0 \leq j \leq N)$ in this auction can be written as

$$P_j = \Pr(n = j|q^*) = \binom{N}{j} q^{*j}(1 - q^*)^{N-j}.$$

3.2. Second-stage: bidding

Following the entry stage using the mixed strategy described above, each active bidder learns his private value v for the lot. It is further assumed that each active bidder does not know the number of active bidders at the time of bidding. Taking into account this uncertainty is more relevant to the real applications. For active bidder i, his expected profit by bidding b_i conditioning on his private value v_i and winning the auction is

$$\begin{aligned} E\pi(b_i, v_i|q^*, N) &= \sum_{j=1}^{N} P_B(n = j)(v_i - b_i) \Pr(b_t \leq b_i, \forall t \neq i) \text{ if } v_i \geq p_0 \\ &= (v_i - b_i) \sum_{j=1}^{N} P_B(n = j) \Pr\left[v_t \leq s^{-1}(b_i|q^*), \forall t \neq i\right] \text{ if } v_i \geq p_0 \\ &= (v_i - b_i) \sum_{j=1}^{N} P_B(n = j) \left\{F\left[s^{-1}(b_i|q^*)\right]\right\}^{j-1} \text{ if } v_i \geq p_0, \end{aligned}$$

where $s(\cdot|q^*)$ is the strictly increasing equilibrium bidding strategy given the equilibrium entry probability q^* from the first stage and

$$P_B(n = j) = \binom{N-1}{j-1} q^{*j-1}(1 - q^*)^{N-j}$$

is the probability of n active bidders from an active bidder's point of view. Note that it is slightly different from P_j defined above.

The Bayesian–Nash equilibrium of this model can be characterized by the following first order condition

$$\frac{\partial \left\{ s(v|q^*) \sum_{j=1}^{N} P_B(n = j)[F(v)]^{j-1} \right\}}{\partial v} = -v \sum_{j=1}^{N} P_B(n = j)(j-1)F(v)^{j-2} f(v). \tag{2}$$

The unique solution to Eq. (2) subject to the boundary condition $s(p_0) = p_0$ is given as follows

$$b = s(v|q^*) = v - \frac{\sum_{j=1}^{N} P_B(n = j) \int_{p_0}^{v} F(x)^{j-1} \mathrm{d}x}{\sum_{j=1}^{N} P_B(n = j) F(v)^{j-1}}. \tag{3}$$

As a result, the equation determining the equilibrium entry probability q^* (1) can be rewritten as

$$\int_{p_0}^{\bar{v}} \sum_{j=1}^{N} P_B(n = j) \int_{p_0}^{v} F(x)^{j-1} \mathrm{d}x f(v) \mathrm{d}v = k. \tag{4}$$

In addition, the density for the number of actual bidders observed in the data n^* conditional on the number of active bidders can be written as

$$\Pr(n^*|n) = \binom{n}{n - n^*} [1 - F(p_0)]^{n^*} F(p_0)^{n - n^*}. \tag{5}$$

This is because each active bidder has the probability $1 - F(p_0)$ of drawing a private value that is higher than the reserve price and then becomes an actual bidder. Furthermore, the density of n^* conditional on the number of potential bidders N can be obtained by integrating out the number of active bidders (which is unobserved in the data) in Eq. (5)

$$\Pr(n^*|N) = \sum_{j=n^*}^{N} P_j \Pr(n^*|j) = \sum_{j=n^*}^{N} P_j \binom{j}{j - n^*} [1 - F(p_0)]^{n^*} F(p_0)^{j - n^*}. \tag{6}$$

3.3. Optimal number of potential bidders

Given the auction model, that is, the IPV information environment, the first-price sealed-bid auction format (the first instrument that the seller can use to influence the auction outcome) and the characteristics of the auction (bidders' private value distribution $F(\cdot)$ and the entry cost k), seller's expected revenue in such an auction depends on the other two instruments that are under seller's control, that is, the reserve price p_0 and the number of potential bidders N.

Proposition 1. *In an auction model with entry and uncertain number of active bidders, the seller's expected revenue is*

$$Y(p_0, N) = \begin{cases} Y_1(p_0, N) \text{ if } \int_{p_0}^{\bar{v}} \int_{p_0}^{v} F(x)^{N-1} \mathrm{d}x f(v) \mathrm{d}v \geq k \\ Y_2(p_0, N) \text{ otherwise} \end{cases}$$

where

$$Y_1(p_0,N)=v_0F(p_0)^N+\int_{p_0}^{\bar{v}}Nf(v)F(v)^{N-1}\left[v-\frac{\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}x}{F(v)^{N-1}}\right]\mathrm{d}v,$$

$$Y_2(p_0,N)=v_0P_0+\sum_{j=1}^{N}P_j\left\{\begin{array}{l} v_0F(p_0)^j \\ +\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}\left[v-\frac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\right]\mathrm{d}v\end{array}\right\}.$$

Furthermore, the total social welfare generated by the auction is

$$S(p_0,N)=\begin{cases}S_1(p_0,N) \text{ if } \int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v\geq k\\ S_2(p_0,N)=Y_2(p_0,N) \text{ otherwise}\end{cases}$$

where

$$S_1(p_0,N)=v_0F(p_0)^N+\int_{p_0}^{\bar{v}}Nf(v)F(v)^{N-1}v\mathrm{d}v-Nk.$$

If a pair of (p_0,N) is chosen such that $\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v\geq k$,[10] then all the potential bidders make a non-negative expected profit by entering into the auction. In such a case, all the potential bidders adopt the pure strategy to enter into the auction in the first stage of the auction game and the second stage of the auction game is reduced to the standard first-price sealed bid auction model without entry. In this case, the seller's expected revenue $Y_1(p_0,N)$ has two terms. The first term captures the case where all the active bidders' private values are less than p_0. In this scenario, the auctioned object goes unsold. The second scenario is that there is at least one active bidder whose private value is above p_0 and hence the auctioned object is sold.

On the other hand, if a pair of (p_0,N) is chosen such that $\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v<k$, then the potential bidders make the entry and bidding decisions as described in the previous subsection. In this case, the seller's expected revenue is $Y_2(p_0,N)$, which has three terms. The first term captures the case where there is no active bidder in the auction and the second term is for the case where there is at least one active bidder. In the latter case, however, there are still two possible scenarios. First, all the active bidders' private values are less than p_0. In this scenario, the auctioned object goes unsold. The second scenario is that there is at least one active bidder whose private value is above p_0 and hence the auctioned object is sold.

The social welfare generated by this auction is simply the sum of the seller's expected revenue and the bidders' expected profits. In the second case where bidders adopt mixed strategy of entry, the bidders' equilibrium profit is always 0 and hence the total expected social welfare generated by the auction equals the seller's expected revenue. This is not the case in the first scenario where bidders adopt pure strategy for their entry decisions.

The optimal auction mechanism from a seller's point of view is characterized by a pair of (p_0,N) that maximizes $Y(p_0,N)$ and the optimal auction mechanism for the society is characterized by another pair of (p_0,N) such that $S(p_0,N)$ is maximized. However, due to the complexity of $Y(p_0,N)$ and $S(p_0,N)$, further characterizing the optimal pairs of (p_0,N) theoretically is difficult. Since the main purpose of this paper is to quantify the cost due to excess market thickness, the optimal number of potential bidders for the auction model under consideration is examined with the restriction that the seller always sets the reserve price at his own value, that is, $p_0=v_0$. The reserve price strategy $p_0=v_0$ is picked for two reasons. First, in empirical studies, the seller's own value of the auctioned object is unobserved and researchers often assume that the observed auction data are generated from an auction mechanism where the seller sets the reserve price at his own value (e.g. Li and Perrigne 2003). As a result, the difference between the outcomes from an auction with $p_0=v_0$ and the optimal number of potential bidders on one hand, and the outcomes observed in the data on the other hand, can be used as a measure for the welfare implications of using the optimal number of potential bidders. Second, as shown in Proposition 2 below, under the constraint that $\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v<k$ (bidders use mixed strategy in the first stage of the auction game), $p_0=v_0$ is actually the optimal reserve price strategy (both to the seller and the society) for any number of potential bidders N. Therefore, $p_0=v_0$, together with an optimal number of potential bidders under $p_0=v_0$ constitute an auction mechanism that may be close to the optimal pairs of (p_0,N) that maximize the seller's expected revenue and the expected social welfare.

Proposition 2. *In an auction model with entry and uncertain number of active bidders, given the assumption that the seller commits to the reserve price strategy $p_0=v_0$, the optimal number of potential bidders from the seller's point of view is N^0, which satisfies $\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N^0-1}\mathrm{d}xf(v)\mathrm{d}v-k\geq 0$ and $\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N^0}\mathrm{d}xf(v)\mathrm{d}v-k<0$. The optimal number of potential bidders N^1 to the society is $N^1=\arg\max_{1\leq N\leq N^0}S_1(v_0,N)$, where $S_1(v_0,N)$ is defined in Proposition 1.*

N^0 is the point of transition from pure to mixed entry strategies. The intuition for the result that N^0 is the optimal number of potential bidders from the seller's point of view is as follows. When $N>N^0$, symmetric equilibrium requires each potential bidder to enter with probability $q^*<1$. Since each entry decision is taken independently, the number of entrants can range from 0 to N. Therefore, the number of active bidders is stochastic and can be too small or too large on occasion, which is unfavorable for the seller. Therefore, the seller's expected revenue will increase monotonically as the number of potential bidders decreases toward N^0. On the other hand, it is not optimal for the seller to suppress the number of potential bidders to be less than N^0. This is because when $N\leq N^0$, the auction game with entry reduces to a standard first-price sealed-bid auction model without entry and it is a well known result that in such a model, the seller's expected revenue increases with the number of potential bidders.

For the expected social welfare, when $N>N^0$, bidders use mixed strategies to decide whether or not to enter into the auction and the expected social welfare equals to seller's expected revenue. Therefore, similar to the seller's expected revenue, it monotonically increases as N decreases toward N^0. But when $N\leq N^0$, the expected social welfare $S_1(v_0, N)$ may not be monotonically increasing in N. This is because the expected social welfare equals the sum of the seller's expected revenue and bidders' expected profits. When $N\leq N^0$, the seller's expected revenue is $Y_1(p_0, N)$, which monotonically increases in N. However, the expected profits for all potential bidders is $N\left[\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N-1}dxf(v)dv-k\right]$, which may not be monotonically increasing in N, depending on the magnitude of entry cost k. As a result, $S_1(v_0, N)$ may not be monotonically increasing in N.

For the optimal number of potential bidders that maximizes the seller's expected revenue, Levin and Smith (1994) obtains a similar result for the auction model with entry and certain number of active bidders. Proposition 2 above generalizes their result to the case of auction model with entry and uncertain number of active bidders. For the optimal number of potential bidders that maximizes the social welfare, the result here is new.

4. Structural estimation

In this section, the corresponding structural models for entry and bidding from the game-theoretic models presented in the previous section are derived. Then, an iterative multi-step maximum likelihood

[10] The left hand side of the inequality is the expected profit for a potential bidder in a standard first-price sealed bid auction without entry and with N potential bidders. It is the special case of the left hand side of Eq. (4) with $P_B(n=N)=1$ and $P_B(n=j)=0\,\forall j\neq N$.

estimation procedure is proposed to jointly estimate the structural models with the data from MDoNR.

The econometrics of structural auction models was first pioneered by Paarsch (1992). The difficulty in the statistical inference of this kind of structural models is the fact that the support of the distribution of the observed dependent variables (the bids in our model) usually depends on the parameters of interest, and hence violates the regularity conditions for the consistency of the maximum likelihood estimator. This can be most clearly seen from Eq. (3), which implies that $b \leq \sum_{j=1}^{N} P_B(n=j)\left[\int_{p_0}^{\bar{v}}(j-1)F(x)^{j-2}f(x)xdx + p_0F(p_0)^{j-1}\right]$. Therefore, the upper support of b depends on the underlying parameters of interest that characterize bidders' private distribution $F(\cdot)$ and the equilibrium entry probability q^*. In order to overcome this difficulty, Donald and Paarsch (1993, 1996) first consider the estimation as a constrained maximum likelihood optimization problem with some constraints becoming binding as the sample size becomes large. More recently, Donald and Paarsch (2002) propose an estimation method using the extreme order statistics, a method that is simpler to implement in practice. But all these estimators are proposed within the standard auction model without endogenous entry and are very difficult to calculate, not mentioning to extend them to the case with endogenous entry.

It is thus not surprising that the recent literature has been mainly focusing on proposing easy-to-implement estimation procedures. Examples include Guerre et al. (2000) for the standard IPV auction model without entry, Li et al. (2002) for affiliated private value (APV) auctions and Krasnokutskaya (2002) for auctions with unobserved heterogeneity. A common feature of these estimators is to use a transformation of the first order condition of the structural auction model so that the unobserved bidders' private values can be recovered directly and hence avoids the parameter dependent support problem.[11] The estimation method proposed here uses a similar idea.

Before moving on, first note that both $P_B(n=j)$ in Eqs. (3) and P_j (6) are binomial probabilities given the number of potential bidders N and the equilibrium entry probability q^*. Since a precise measure for the number of potential bidders is not available, a Poisson distribution is used to approximate the binomial probabilities, following Bajari and Hortaçsu (2003) and Athey et al. (2004). Specifically, the number of active bidders is assumed to follow a Poisson distribution with mean λ. Then, noticing the fact that

$$1 = \sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \Big/ [1-e^{-\lambda}] = \sum_{j=1}^{N} P_B(n=j)$$

by definition, the terms

$$\sum_{j=1}^{N} P_B(n=j)\int_{p_0}^{v} F(x)^{j-1}\mathrm{d}x$$

and

$$\sum_{j=1}^{N} P_B(n=j)F(v)^{j-1}$$

in Eq. (3) are approximated using

$$\sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \Big/ [1-e^{-\lambda}]\int_{p_0}^{v} F(x)^{j-1}\mathrm{d}x$$

and

$$\sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \Big/ [1-e^{-\lambda}]F(v)^{j-1},$$

respectively. Also note that

$$\sum_{j=n^*}^{N} P_j = \sum_{j=n^*}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} = 1-\Pr(n<n^*)$$

by definition. Therefore, the term

$$\sum_{j=n^*}^{N} P_j \binom{j}{j-n^*}[1-F(p_0)]^{n^*}F(p_0)^{j-n^*}$$

in Eq. (6) is approximated using

$$\sum_{j=n^*}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}\binom{j}{j-n^*}[1-F(p_0)]^{n^*}F(p_0)^{j-n^*}.$$

As a result, Eqs. (3) and (6) can be rewritten as

$$\begin{aligned} b &= v - \frac{\sum_{j=1}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}/[1-e^{-\lambda}]\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}/[1-e^{-\lambda}]F(v)^{j-1}}, v \geq p_0 \\ &= v - \frac{\sum_{j=1}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}F(v)^{j-1}}, v \geq p_0 \end{aligned} \tag{7}$$

and

$$\Pr(n^*) = \sum_{j=n^*}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}\binom{j}{j-n^*}[1-F(p_0)]^{n^*}F(p_0)^{j-n^*}. \tag{8}$$

In principal, after making specific parametric assumptions on λ and $F(\cdot)$, a maximum likelihood estimator based on the density function for the number of actual bidders (8) can consistently estimate the model primitives λ and $F(\cdot)$. However, this approach may not be efficient because it only uses data on the number of actual bidders and does not use the data on the bids. Alternatively, one can form moment conditions on the bids using Eq. (7) and then combine such moment conditions with the moment conditions based on Eq. (8) to estimate the model primitives more efficiently using the simulated method of moments (MSM). Li (2005) takes such an approach for an auction model with entry and certain number of active bidders. However, that approach has two shortcomings. First, the MSM approach is a moment based approach, which is known to be less efficient than a likelihood based approach proposed below. Second, as mentioned in Li (2005), to implement the MSM approach, those auctions that receive no bids must be excluded from the sample (which accounts for more than 5% of the data in our sample). This is because for those observations, the sample moment condition for the bids is not defined.

To overcome these difficulties, an iterative multi-step maximum likelihood estimation procedure is proposed[12]:

1. Using data on the number of actual bidders alone and Eq. (8) to get an initial consistent estimator of λ and $F(\cdot)$, denoted as λ_1 and $F_1(\cdot)$ using the maximum likelihood estimation method.
2. Using λ_1 and $F_1(\cdot)$ and the observed bids b to recover private values v using Eq. (7) for all the observed bids. According to the model, the

[11] An alternative way to overcome the parameter support dependent problem in auction models is to use moment based estimation methods. Examples include Laffont et al. (1995) and Li and Vuong (1997) using the simulated nonlinear least squares method for standard IPV auctions without entry.

[12] Appendix B details how the standard errors are computed.

recovered private values v are random variables from the truncated density $\frac{f(\cdot)}{1-F(p_0)}$.

3. Using v from the second step and its corresponding density $\frac{f(\cdot)}{1-F(p_0)}$ to obtain the second estimate of $F(\cdot)$, denoted as $F_2(\cdot)$ through maximum likelihood estimation. Fixing $F(\cdot)$ at $F_2(\cdot)$ and using data on the number of actual bidders alone, maximize the log likelihood function based on Eq. (8) with respect to λ alone to get the second estimator for λ, denoted as λ_2.
4. Repeat steps 2 and 3 until convergence.

Using the iterative multi-step maximum likelihood estimation procedure described above for the structural auction model with endogenous entry considered in this paper enjoys several advantages. First, it is computationally straightforward. It breaks a complex estimation problem into three very simple tasks. The first step is a maximum likelihood estimation involving only data on the number of actual bidders. The second step is a calculation step to recover the private values. And the final step is a maximum likelihood estimator with a simple likelihood function, that is, the truncated density for private values. Second, it is more efficient as both the data on the number of actual bidders and the observed bids are used and it is a likelihood based approach rather than a moment based approach. Third, it does not suffer the parameter-dependent problem as the maximum likelihood approach based upon the derived density for the bids. Fourth, unlike the MSM approach discussed above, this approach can accommodate the observations with 0 number of actual bidders, which will increase the efficiency of the estimates.[13]

4.1. Specification

In an econometric investigation, one often considers more than one auction, and the statistical inference is usually based on the assumption that the number of auctions approaches infinity. Therefore, possible heterogeneity across auctions needs to be taken into account. This issue can be addressed by modelling the distribution of the private costs to depend on the heterogeneity of auctioned objects and thus to vary across auctions. Specifically, let $F_\ell(\cdot)$ denote the distribution of private costs for the ℓ-th auction, $\ell=1,\ldots,L$, where L is the number of auctions. Assume that $F_\ell=F(\cdot|x_\ell,\beta)$, where x_ℓ is a vector of variables that represents the observed auction heterogeneity, affecting bidders' costs, and β is an unknown parameter vector in $B\subset\mathbb{R}^{m^1}$. Let $f(\cdot|x_\ell,\beta)$ denote the corresponding density for bidders' private costs. In this paper, the density of bidders' private values for the timber is parameterized as follows

$$f(v|x)=\frac{1}{\exp(x\beta)}\exp\left[-\frac{1}{\exp(x\beta)}v\right], \tag{9}$$

for $v\in(0,\infty)$.[14]

To make the above iterative estimation procedure work, the mean of the Poisson density for the number of active bidders, λ_ℓ, also needs to be specified. From Eq. (4), it is clear that the equilibrium entry probability q_ℓ^* is a function of the entry cost k_ℓ, which itself is likely to be correlated with auction heterogeneity x_ℓ. Also, q_ℓ^* depends on the number of potential bidders N_ℓ and covariates x_ℓ through the dependence on F and f. Taking these considerations into account, λ_ℓ (an approximation for $N_\ell q_\ell^*$) is specified to be

$$\lambda_\ell=\exp\left(\gamma_0\tilde{N}_\ell+x_\ell\gamma\right) \tag{10}$$

where $\tilde{N}_\ell$ is the rough measure for the number of potential bidders.

Table 5
Reduced form estimation for the distribution of bids.

Variable	With unobserved heterogeneity		Without unobserved heterogeneity	
	Coefficient	z-statistic	Coefficient	z-statistic
β (coefficients in the scale parameter)				
log(Reserve)	0.9901	178.91	0.9216	8.39
Acre	0.0061	0.68	0.0634	3.15
Range	−0.0004	−0.01	−0.0283	−0.24
Years	0.0358	2.10	0.0930	2.40
Potential	0.0003	0.31	−0.0030	−2.12
Actual	0.0204	9.50	0.0254	7.21
Constant	0.0652	1.05	0.8188	6.28
δ (coefficients in the shape parameter)				
Actual	−0.0741	−5.81	−0.0044	−0.39
Constant	3.4058	25.85	1.4781	24.47
θ (variance of the unobserved heterogeneity)				
	3.9763	7.82		
Log Likelihood	−11,993.16		−12,377.21	

Models are specified in Eq. (11).

4.2. Unobserved heterogeneity

One caveat of the proposed estimator is that it does not allow the presence of unobserved auction heterogeneity as in Athey et al. (2004) and Li and Zheng (2008). In principle, the iterative estimator proposed in this paper can be adapted to accommodate the presence of unobserved auction heterogeneity. But doing so would cause the estimator to suffer from the incidental-parameter problem (Neyman and Scott, 1948), as in nonlinear panel data models with fixed effects. This in turn would lead to inconsistent estimates, both for the incidental parameters (unobserved auction heterogeneity) and the structural parameters of interest. Solving this problem is left for future research.

Here, an informal check is conducted to evaluate whether a specification with unobserved auction heterogeneity is more appropriate for the current dataset. Athey et al. (2004) estimates a reduced form bid distribution using the following specification

$$G\left(b|x,n^*,\tilde{N},u\right)=1-\exp\left[-u\left(\frac{b}{\delta}\right)^p\right] \tag{11}$$

where $\delta=\exp\,(x\beta_x+n^*\beta_n+\tilde{N}\beta_N)$ and $p=\exp\,(\delta_0+\delta_n n^*)$. u is the unobserved auction heterogeneity and is assumed to follow the gamma distribution with mean 1 and variance θ. This specification and the same specification without the unobserved auction heterogeneity (u term in Eq. (11)) are estimated. Results are collected in Table 5. It shows that eight out of the nine parameters estimates share the same sign across the two specifications. The magnitudes of the estimates are also similar. On the other hand, the estimate for the variance of the unobserved auction heterogeneity is significant. In addition, BIC and AIC values based on the log likelihood values also indicate the model with unobserved auction heterogeneity is the preferred specification, though the differences are small.

Next, to further assess which specification fits the data better, the in-sample mean squared error of predictions (MSEP) for the bids is computed for both specifications, using the estimated parameters. The MSEP for the specification with unobserved auction heterogeneity

[13] It is worth noting that in a recent paper, Li and Zheng (2008) propose a fully structural estimation method to estimate a similar auction model with endogenous entry and uncertain number of active bidders for low-bid highway mowing auctions. That approach requires the observation of a precise measure of the number of potential bidders and hence cannot be used here because such a measure is not observed in this dataset.

[14] The literature on parametric estimation of auction models has usually used extreme value distributions such as Weibull and exponential to specify private values distributions. See, e.g., Paarsch (1992, 1997), Donald and Paarsch (1993, 1996), Haile (2001), Athey et al. (2004).

Table 6
Estimates of structural models.

Variable	Coefficient	t-statistic
β (coefficients in mean of the private value distribution)		
log(Reserve)	0.7453	16.22
Years	0.3383	2.20
Acre/100	0.1505	1.86
Range	−0.6386	−1.33
Constant	1.3792	2.68
γ (coefficients in mean of the number of active bidders)		
log(Reserve)	0.5691	15.32
Years	−0.4440	−4.71
Acre/100	−0.4222	−6.34
Range	0.0836	0.21
Constant	−1.9919	−4.96
γ_0	0.0323	5.88

Models are specified in Eqs. (7), (8), (9) and (10).

is $4.0066*10^8$ and the MSEP for the specification without unobserved auction heterogeneity is $2.8002*10^8$. As a result, the MSEP results are clearly in favor of the specification without unobserved auction heterogeneity.

Mixed evidence regarding which specification (with and without unobserved heterogeneity) is better, together with the facts that the exogenous variables have already captured most of the auction heterogeneity as shown in the Section 2[15] and the estimates are similar across the two specifications as shown in Table 5, lead to the conclusion that the unobserved auction heterogeneity is less of a concern here.

5. Results

5.1. Parameter estimates

The estimation results from the iterative multi-step maximum likelihood estimator are summarized in Table 6. First, note that in the private value distribution, the variable log(Reserve) has a large positive and significant effect. For the private value distribution, the estimated coefficient is 0.7453, which indicates that the reserve price is a good estimate for the values of the auctioned timber lots and bidders use this information to determine their own private values for the timber lot. A1 unit increase in the log(Reserve) will increase the mean of bidders' private value by 2.11 times. Log(Reserve) also has a positive and significant effect on the mean of the number of active bidders, λ. This is reasonable since the bidders may enjoy some economies of scale in the bidding preparation process. Hence, a timber lot with bigger value implies a relatively low entry cost on a per unit worth of timber basis and more bidders are induced to enter into such auctions.

Second, it is worth noting that the number of potential bidders has a significant positive effect on the mean of the number of active bidders. When the number of potential bidders increases, it has two effects on the mean of the number of active bidders. First, everything else equal, when there are more potential bidders, bidders' expected winning probability decreases and hence less bidders enter into the auction. Second, as the mean of the number of active bidders equals the product of the number of potential bidders and the entry probability, holding the equilibrium entry probability fixed, an increase in the number of potential bidders has a positi ve effect on the mean of the number of active bidders. A positive coefficient here indicates that the second effect dominates the first effect in the data.

Third, very interestingly, the number of years of allowed harvest time and the number of acres of the lot have opposite effects on the bidders' private value distribution and the mean of the number of active bidders. For the mean of the number of active bidders, the effects are negative because these two variables are likely to be associated with geographic complexity of the timber lot and hence active bidders need to incur higher costs to cruise the lot to develop private values. Expecting a higher entry cost, less potential bidders choose to enter the auction, leading to a lower number of active bidders. On the other hand, once they have already incurred the sunk entry cost and entered the auction, active bidders may place positive values on these features of the lot. This is because a longer harvest time period makes the winner's work schedule more flexible and a larger lot (after controlling the value of the lot) makes the harvest job easier.

Furthermore, we use the structural estimates to recover values for several variables of economic interest. Table 7 reports the percentiles for the entry cost k, the mean of the private values $\exp(x\beta)$, the winner's payoff defined as the winner's private value less the sum of the winning bid and the entry cost and the winner's information rent defined as the ratio between the winner's payoff and the winning bid. Several interesting results are worth mentioning. First, the median ratio between the entry cost and the mean of the private values is about 5.61%, meaning that the entry cost accounts for about 5.61% of the private values. This finding indicates that entry cost is a nontrivial part of bidders' decision making process and hence a theoretical model or an empirical analysis that ignore the entry effect may lead to misleading policy recommendations.

Second, it is interesting to note that in contrast to results based on standard auction models without entry, not all winners have a positive ex post payoff from the auction. When entry is present, the theoretical model above predicts that in equilibrium, a potential bidder's expected payoff is 0. Ex post, the winner's payoff tends to be positive as the winner is the active bidder with the highest private value, but it can also be negative, depending on the private value they draw after entering into the auction and the entry cost they incur to enter into the auction. This is confirmed by our estimates of the winner's payoff and winner's information rent as some winners make a positive payoff and others have a negative payoff. Table 7 shows that the median winner's payoff from the auctions is \$6,271.70 and the median information rent is 22%.

5.2. Quantifying the cost of excess market thickness

With the estimates on model primitives, I now can examine whether there is excess market thickness in the timber auction markets and further quantify the cost, both to the seller and to the society, if there is such excess market thickness.

To do so, I first need to get an estimate of the optimal number of potential bidders for each auction. Proposition 2 says that when the seller commits to the reserve price strategy $p_0 = v_0$, the optimal

Table 7
Percentiles of entry cost, mean of private values, winner payoff and winner's information rent.

Percentile	k (\$)	$\exp(x_t\beta)$ (\$)	$v_1 - b_1 - k$ (\$)	$\frac{v_1-b_1-k}{b_1}$
Min	95.43	989.30	−1783.66	−0.09
10	402.62	3878.80	468.41	0.08
25	530.12	6812.48	2959.81	0.17
50	725.13	12,931.57	6271.70	0.22
75	1029.03	22,254.18	14,459.79	0.33
90	1591.72	33,583.35	29,416.91	0.55
Max	4079.98	93,333.08	176,661.21	1.17

k: entry cost; $\exp(x_t\beta)$: mean of private values; $v_1 - b_1 - k$: winner payoff; $\frac{v_1-b_1-k}{b_1}$: winner's information rent.

[15] In a recent paper, Roberts (2008) shows both theoretically and empirically why seller's reserve price has "great ability" to control for unobserved auction heterogeneity and help "soak up" variation in bids. The seller's reserve price is included as one of the covariates throughout the analysis.

number of potential bidders from the seller's point of view is N^0, which satisfies

$$\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N^0-1}\mathrm{d}xf(v)\mathrm{d}v-k\geq 0 \tag{12}$$

and

$$\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N^0}\mathrm{d}xf(v)\mathrm{d}v-k<0. \tag{13}$$

Using the estimates of the structural primitives of the model, I can recover an estimate for N^0 easily. If the estimated optimal N^0 is different from the number of potential bidders in the data, then the current auction mechanism is not optimal. In this case, the seller's cost due to excess market thickness can be quantified as the difference between the predicted winning bid if the auction is restricted to have at most N^0 number of potential bidders and the observed winning bid. To get the predicted winning bid, I first use the winning bid and Eq. (3) to recover the private value of the winner. And then what the winner would bid under the new auction mechanism is predicted. In the new auction with N^0 number of potential bidders, bidders have a positive equilibrium expected profit from entering into the auction and hence all the bidders enter into the auction and become active bidders. In such a case, the equilibrium bidding strategy becomes

$$s(v)=v-\frac{\int_{p_0}^{v}F(x)^{N^0-1}\mathrm{d}x}{F(v)^{N^0-1}}. \tag{14}$$

Results are reported in Table 8. The ratio between the median number of optimal number of potential bidders and the median of the estimated mean of the number of active bidders λ is 0.93. This indicates there exists excess market thickness in timber sale auctions as the unobserved number of potential bidders $N=\frac{\lambda}{q^*}$ is larger than λ by construction. Switching to the auction mechanism with the optimal number of potential bidders, the median winning bid is \$45.46, or 0.15% higher, indicating the loss to the seller due to the excess market thickness is minimal in this market.[16]

Next, the social cost of excess market thickness is quantified. In the observed data, the social welfare generated by an auction equals the difference between the winner's private value and the seller's own value (the social gain of the transaction), minus the entry costs incurred by all the active bidders. Using the estimated model primitives, the winner's private value can be recovered using the winning bid. The seller's own value is assumed to be revealed by the public reserve price. Finally, as the number of active bidders, n, is not observed, it is approximated using the estimated mean of the number of active bidders λ. When the social optimal number of potential bidders N^1 is used, all the potential bidders enter into the auction. In this case, the first part of the social welfare, that is, the difference between the winner's private value and the seller's own value, is unchanged. But the entry costs incurred are N^1k. As a result, the gain in the social welfare by adopting the social optimal number of potential bidders is $(\lambda-N^1)k$. Results are reported in Table 8. In general, the social optimal number of potential bidders should be smaller than the optimal number of potential bidders from the seller's point of view. This is because the seller does not take into account the loss in terms of entry cost, while the society does. Due to the presence of entry cost, too many potential bidders and hence too many active bidders are very costly to the society. However, in the particular sample of timber auctions studied in this paper, the estimated entry costs are small and N^1 turns out to be equal to N^0 for each auction. After the switch, the median social welfare gain is \$1006.29, which accounts for 3.39% of the observed winning bid. This indicates that the welfare loss due to sub-optimal number of potential bidders is moderate in this market.

Table 8
Percentiles of optimal number of potential bidders and cost due to excess market thickness.

Percentile	λ	N^0	N^1	Cost to seller (\$)	Cost to society (\$)
Min	2.71	2	2	−1755.00	90.80
10	8.48	8	8	−327.11	508.48
25	12.06	11	11	−96.06	663.68
50	16.12	15	15	45.46	1006.29
75	22.95	22	22	149.27	1475.21
90	30.00	28	28	255.36	2161.61
Max	47.00	37	37	1776.06	6318.56

λ: estimated mean of the number of active bidders; N^0: optimal number of potential bidders to the seller; N^1: optimal number of potential bidders to the society.

6. Conclusion

In this paper, a computationally easy method is proposed to estimate the optimal number of potential bidders in timber sale auctions with endogenous entry and an uncertain number of active bidders. Estimates are then used to quantify the cost of excess thickness in the Michigan timber sale market. Switching to the auction mechanisms with the optimal number of potential bidders leads to only small increase in both the median winning bid and the median social welfare. Hence, it can be concluded that the welfare loss due to the excess market thickness is moderate in this market.

Appendix A. Proofs

Proof of Proposition 1. When the number of potential bidders N is chosen such that entering into the auction always yields a positive expected payoff for a potential bidder, the equilibrium entry strategy for potential bidders is a pure strategy, that is, enter into the auction with probability 1. This reduces the model to a standard first-price high bid auction model in the IPV framework and bidders' equilibrium bidding strategy at the bidding stage is

$$b=s(v)=v-\frac{\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}x}{F(v)^{N-1}}\ \text{if } v\geq p_0$$

and the expected gain from entering and bidding in the auction for one bidder is

$$\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v-k.$$

As a result, the total expected gains for all the bidders is

$$\int_{p_0}^{\bar{v}}\int_{p_0}^{v}NF(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v-Nk.$$

Therefore, when N is chosen such that $\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v\geq k$, all potential bidders enter into the auction. In this case, the seller's expected revenue can be written as

$$\begin{aligned}Y_1(p_0,N)&=v_0F(p_0)^N+\left(1-F(p_0)^N\right)\int_{p_0}^{\bar{v}}\frac{Nf(v)F(v)^{N-1}}{1-F(p_0)^N}s(v)\mathrm{d}v\\&=v_0F(p_0)^N+\int_{p_0}^{\bar{v}}Nf(v)F(v)^{N-1}\left[v-\frac{\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}x}{F(v)^{N-1}}\right]\mathrm{d}v\end{aligned}$$

[16] Note that for some auctions, switching to the optimal number of potential bidders N^0 results in a lower winning bid. This is because N^0 is defined as the number of potential bidders that maximizes the seller's ex ante expected revenue before bidders' private values are realized. Ex post, whether the seller can gain from the switch also depends on the realized private value of the winner. But a measure like the median is informative on the ex ante gain.

where the first term captures the case where all active bidders' private values are less than p_0. In this scenario, the auctioned object goes unsold. The second scenario is that there is at least one active bidder whose private value is above p_0 and hence the auctioned object is sold. Also, the expected social welfare is the sum of the seller's expected revenue and the total expected gains for all the bidders

$$S_1(p_0,N)=Y_1(p_0,N)+\int_{p_0}^{\bar{v}}\int_{p_0}^{v}NF(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v-Nk$$
$$=v_0F(p_0)^N+\int_{p_0}^{\bar{v}}Nf(v)F(v)^{N-1}v\mathrm{d}v-Nk.$$

On the other hand, when N is chosen such that $\int_{p_0}^{\bar{v}}\int_{p_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v<k$, potential bidders adopt the mixed strategy to determine whether or not to enter into the auction. In this case, all the potential bidders make 0 expected profit and the seller's expected revenue can be written as

$$Y_2(p_0,N)=v_0p_0+\sum_{j=1}^{N}p_j\left\{\begin{array}{l}v_0F(p_0)^j\\+\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}\left[v-\dfrac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\right]\mathrm{d}v\end{array}\right\}.$$

The first term captures the case where there is no active bidder in the auction and the second term is for the case where there is at least one active bidder. In the latter case, however, there are still two possible scenarios. First, all the active bidders' private values are less than p_0. In this scenario, the auctioned object goes unsold. The second scenario is that there is at least one active bidder whose private value is above p_0 and hence the auctioned object is sold. Again, the expected social welfare is the sum of the seller's expected revenue and the total expected gains for all the bidders (0 here). Hence $S_2(p_0,N)=Y_2(p_0,N)$.

Proof of Proposition 2. The assumption that the seller commits to the reserve price strategy $p_0=v_0$ is maintained throughout the proof. Hence, p_0 is replaced by v_0 in all the formulas. First, the following is proved: when N is chosen such that bidders adopt a mixed strategy to decide whether or not to enter into the auction in the first stage, $Y_2(v_0, N)=S_2(v_0, N)$ monotonically increases as N decreases toward N^0.

From Lemma 1 below, $Y_2(v_0, N)$ can be rewritten as

$$Y_2(v_0,N)=\sum_{j=0}^{N}P_jv_0F(v_0)^j+\sum_{j=1}^{N}P_j\int_{v_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v-Nq^*k.$$

Denote the q_N^* as the equilibrium entry probability when the number of potential bidders is N. Then, holding the equilibrium entry probability at q_N^* and decreasing the number of potential bidders by 1, yields the difference in the seller's expected revenue as

$$\begin{aligned}\Delta Y_2&=Y_2(v_0,N)-Y_2(v_0,N-1)\\&=\sum_{j=0}^{N}P_jv_0F(v_0)^j-\sum_{j=0}^{N-1}\frac{(N-1)!}{(N-1-j)!j!}q_N^{*j}(1-q_N^*)^{N-1-j}v_0F(v_0)^j\\&\quad+\sum_{j=1}^{N}P_j\int_{v_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\\&\quad-\sum_{j=1}^{N-1}\frac{(N-1)!}{(N-1-j)!j!}q_N^{*j}(1-q_N^*)^{N-1-j}\int_{v_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v-q_N^*k\\&=\sum_{j=0}^{N}P_jv_0F(v_0)^j\frac{(j-Nq_N^*)}{N(1-q_N^*)}+\sum_{j=1}^{N}P_j\int_{v_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\frac{(j-Nq_N^*)}{N(1-q_N^*)}-q_N^*k.\end{aligned}$$

On the other hand,

$$\left.\frac{\partial Y_2(v_0,N)}{\partial q^*}\right|_{q^*=q_N^*}=\sum_{j=0}^{N}P_jv_0F(v_0)^j\frac{(j-Nq_N^*)}{q_N^*(1-q_N^*)}+\sum_{j=1}^{N}P_j\int_{v_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\frac{(j-Nq_N^*)}{q_N^*(1-q_N^*)}-Nk.$$

As a result,

$$\Delta Y_2=\frac{q_N^*}{N}\left.\frac{\partial Y_2(v_0,N)}{\partial q^*}\right|_{q^*=q_N^*}=0$$

where the second equality follows from Lemma 2 below. Thus, reducing the number of potential bidders by 1 while holding the equilibrium entry probability at q_N^* leaves the seller's expected revenue unchanged. If the constraint on q_N^* is relaxed now, in equilibrium, the new equilibrium entry probability will be q_{N-1}^* as determined by the entry equation

$$\int_{v_0}^{\bar{v}}\sum_{j=1}^{N-1}P_B(n=j)\int_{v_0}^{v}F(x)^{j-1}\mathrm{d}xf(v)\mathrm{d}v=k.$$

Since in Lemma 2 below, it is proved that for any number of potential bidders such that bidders adopt the mixed entry strategy at the first stage of the auction game, the reserve price strategy $p_0=v_0$ and its corresponding equilibrium entry probability q^* maximizes the seller's expected revenue, it follows that q_{N-1}^* maximizes $Y_2(v_0, N-1)$. This implies that $\Delta Y_2<0$ when the constraint on q_N^* above is relaxed. This further implies that $Y_2(v_0, N)$ monotonically increases as N decreases toward N^0. In addition, since in this case, $Y_2(v_0, N)=S_2(v_0, N)$, $S_2(v_0, N)$ has the same property.

On the other hand, when N is such that $\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v-k\geq 0$, bidders adopt the pure strategy to determine their entry behavior at the first stage of entry. In this case, the seller's expected revenue is $Y_1(v_0, N)$. It is a well known theoretical result that $Y_1(v_0, N)$ increases monotonically in N as long as $\int_{v_0}^{\bar{v}}\int_{v_0}^{v}F(x)^{N-1}\mathrm{d}xf(v)\mathrm{d}v-k\geq 0$. The proof is omitted here for brevity. In conclusion, given the seller's reserve price strategy $p_0=v_0$, N^0 is the optimal number of potential bidders that maximizes the seller's expected revenue.

Regarding the social welfare, $S_1(v_0, N)$ is not a monotone function of N. Hence, the optimal number of potential bidders for the society is less than or equal to N^0 and maximizes $S_1(v_0, N)$, that is, $N^1=\operatorname{argmax}_{1\leq N\leq N^0}S_1(v_0, N)$. This completes the proof.

Lemma 1.

$$Y_2(p_0,N)=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v-Nq^*k.$$

Proof of Lemma 1.

$$\begin{aligned}Y_2(p_0,N)&=v_0P_0+\sum_{j=1}^{N}P_j\left\{\begin{array}{l}v_0F(p_0)^j\\+\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}\left[v-\dfrac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\right]\mathrm{d}v\end{array}\right\}\\&=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\\&\quad-\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}\frac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\mathrm{d}v\\&=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\\&\quad-\int_{p_0}^{\bar{v}}\sum_{j=1}^{N}\frac{N!}{(N-j)!j!}q^{*j}(1-q^*)^{N-j}jf(v)F(v)^{j-1}\frac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\mathrm{d}v\\&=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\\&\quad-Nq^*\int_{p_0}^{\bar{v}}\sum_{j=1}^{N}P_B(n=j)f(v)F(v)^{j-1}\frac{\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}x}{\sum_{j=1}^{N}P_B(n=j)F(v)^{j-1}}\mathrm{d}v\\&=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v\\&\quad-Nq^*\int_{p_0}^{\bar{v}}\sum_{j=1}^{N}P_B(n=j)\int_{p_0}^{v}F(x)^{j-1}\mathrm{d}xf(v)\mathrm{d}v\\&=\sum_{j=0}^{N}P_jv_0F(p_0)^j+\sum_{j=1}^{N}P_j\int_{p_0}^{\bar{v}}jf(v)F(v)^{j-1}v\mathrm{d}v-Nq^*k\end{aligned}$$

where the last equality follows from Eq. (4).

Lemma 2. $\frac{\partial Y_2(v_0,N)}{\partial q^*} = 0.$

Proof of Lemma 2.

$$\begin{aligned}
\frac{\partial Y_2(v_0,N)}{\partial q^*} &= v_0\sum_{j=0}^{N}\frac{\partial P_j}{\partial q^*}F(v_0)^j + \sum_{j=1}^{N}\frac{\partial P_j}{\partial q^*}j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v - Nk \\
&= v_0\sum_{j=0}^{N}\frac{\partial P_j}{\partial q^*}F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\left[\begin{array}{c}\binom{N}{j}j(q^*)^{j-1}(1-q^*)^{N-j} \\ -\binom{N}{j}(N-j)(q^*)^j(1-q^*)^{N-j-1}\end{array}\right]j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v - Nk \\
&= v_0\sum_{j=0}^{N}\frac{\partial P_j}{\partial q^*}F(v_0)^j \\
&\quad + \left[\begin{array}{c}\sum_{j=1}^{N}P_j j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v(j-Nq^*) \\ -q^*Nk(1-q^*)\end{array}\right]\Big/[q^*(1-q^*)].
\end{aligned} \tag{A.1}$$

Lemma 3 below shows that

$$q^*Nk = \sum_{j=1}^{N}P_j j\left\{j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v - (j-1)\int_{v_0}^{\bar v}F(v)^{j-2}f(v)\mathrm{d}v - v_0[1-F(v_0)]F(v_0)^{j-1}\right\}. \tag{A.2}$$

Using Eq. (A.2), the second term in the last line of Eq. (A.1) becomes

$$\begin{aligned}
&\left[\sum_{j=1}^{N}P_j j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v(j-Nq^*) - q^*Nk(1-q^*)\right]\Big/[q^*(1-q^*)] \\
&= \frac{\left\{\begin{array}{c}\sum_{j=1}^{N}P_j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v(j-Nq^*) \\ -(1-q^*)\sum_{j=1}^{N}P_j j\left\{\begin{array}{c}j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v \\ -(j-1)\int_{v_0}^{\bar v}F(v)^{j-2}f(v)\mathrm{d}v - v_0[1-F(v_0)]F(v_0)^{j-1}\end{array}\right\}\end{array}\right\}}{q^*(1-q^*)}.
\end{aligned} \tag{A.3}$$

Denote

$$T_j(v_0) = 1 - F(v_0)^{j-1} \text{ and } V_j = \frac{j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v}{1-F(v_0)^{j-1}},$$

Eq. (A.3) can be rewritten as

$$\begin{aligned}
&\left[\sum_{j=1}^{N}P_j j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v(j-Nq^*) - q^*Nk(1-q^*)\right]\Big/[q^*(1-q^*)] \\
&= \frac{\sum_{j=1}^{N}P_jT_j(v_0)V_j(j-Nq^*) - (1-q^*)\sum_{j=1}^{N}P_j\left[\begin{array}{c}j(T_j(v_0)V_j - T_{j-1}(v_0)V_{j-1}) \\ -jv_0(1-F(v_0))F(v_0)^{j-1}\end{array}\right]}{q^*(1-q^*)} \\
&= \frac{\left\{\begin{array}{c}\sum_{j=1}^{N}P_jT_{j-1}(v_0)jV_{j-1} - \sum_{j=1}^{N}P_jT_j(v_0)V_jq^*(N-j)\Big/(1-q^*) \\ +\sum_{j=1}^{N}P_jjv_0(1-F(v_0))F(v_0)^{j-1}\end{array}\right\}}{q^*}. \\
&= \frac{\left\{\begin{array}{c}\sum_{j=1}^{N}P_jT_{j-1}(v_0)jV_{j-1} - \sum_{j=1}^{N-1}P_{j+1}T_j(v_0)(j+1)V_j \\ +\sum_{j=1}^{N}P_jjv_0(1-F(v_0))F(v_0)^{j-1}\end{array}\right\}}{q^*} \\
&= \frac{\sum_{j=1}^{N}P_jjv_0(1-F(v_0))F(v_0)^{j-1}}{q^*} \text{ since } V_0 = 0
\end{aligned} \tag{A.4}$$

Therefore, plugging Eq. (A.4) in Eq. (A.1) yields

$$\begin{aligned}
\frac{\partial Y(v_0,N)}{\partial q^*} &= v_0\sum_{j=0}^{N}\frac{\partial P_j}{\partial q^*}F(v_0)^j + \frac{\sum_{j=i}^{N}P_jjv_0(1-F(v_0))F(v_0)^{j-1}}{q^*} \\
&= \sum_{j=0}^{N}\left[\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j} - \binom{N}{j}(N-j)q^{*j}(1-q^*)^{N-j-1}\right]v_0F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j}(1-F(v_0))v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 + \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^j \\
&\quad - \sum_{j=1}^{N}\binom{N}{j}(N-j)q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q)^{N-j}v_0F(v_0)^{j-1} \\
&\quad - \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^j \\
&= -N(1-q^*)^{N-1}v_0 - \sum_{j=1}^{N}\binom{N}{j}(N-j)q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 - \sum_{j=1}^{N-1}\binom{N}{j}(N-j)q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\binom{N}{j}jq^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 - \sum_{j=1}^{N-1}\frac{N!}{(N-j-1)!j!}q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + \sum_{j=1}^{N}\frac{N!}{(N-j)!(j-1)!}q^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 - N\sum_{j=1}^{N-1}\frac{(N-1)!}{(N-j-1)!j!}q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + N\sum_{j=1}^{N}\frac{(N-1)!}{(N-j)!(j-1)!}q^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 - N\sum_{j=1}^{N-1}\binom{N-1}{j}q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + N\sum_{j=1}^{N}\binom{N-1}{j-1}q^{*j-1}(1-q^*)^{N-j}v_0F(v_0)^{j-1} \\
&= -N(1-q^*)^{N-1}v_0 - N\sum_{j=1}^{N-1}\binom{N-1}{j}q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&\quad + N\sum_{j=0}^{N-1}\binom{N-1}{j}q^{*j}(1-q^*)^{N-j-1}v_0F(v_0)^j \\
&= -N(1-q^*)^{N-1}v_0 + N(1-q^*)^{N-1}v_0 = 0
\end{aligned}$$

This completes the proof.

Lemma 3.

$$q^*Nk = \sum_{j=1}^{N}P_j j\left\{\begin{array}{c}j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v \\ -(j-1)\int_{v_0}^{\bar v}F(v)^{j-2}f(v)\mathrm{d}v - v_0[1-F(v_0)]F(v_0)^{j-1}\end{array}\right\}.$$

Proof of Lemma 3. Eq. (4) implies

$$\begin{aligned}
q^*Nk &= \sum_{j=1}^{N}P_j j\int_{v_0}^{\bar v}\int_{v_0}^{v}F(x)^{j-1}\mathrm{d}xf(v)\mathrm{d}v \\
&= \sum_{j=1}^{N}P_j j\left[\int_{v_0}^{v}F(x)^{j-1}F(v)\Big|_{v_0}^{\bar v} - \int_{v_0}^{\bar v}F(v)^j\mathrm{d}v\right] \\
&= \sum_{j=1}^{N}P_j j\left[\int_{v_0}^{\bar v}F(x)^{j-1}\mathrm{d}x - \int_{v_0}^{\bar v}F(v)^j\mathrm{d}v\right] \\
&= \sum_{j=1}^{N}P_j j\left[F(x)^{j-1}x\Big|_{v_0}^{\bar v} - \int_{v_0}^{\bar v}x(j-1)F(x)^{j-2}f(x)\mathrm{d}x - F(v)^jv\Big|_{v_0}^{\bar v} + \int_{v_0}^{\bar v}vjF(v)^{j-1}f(v)\mathrm{d}v\right] \\
&= \sum_{j=1}^{N}P_j j\left\{j\int_{v_0}^{\bar v}F(v)^{j-1}f(v)\mathrm{d}v - (j-1)\int_{v_0}^{\bar v}F(v)^{j-2}f(v)\mathrm{d}v - v_0[1-F(v_0)]F(v_0)^{j-1}\right\}
\end{aligned}$$

Appendix B. Computing standard errors

This appendix details how the standard errors for the estimates from the iterative multi-step maximum likelihood estimation procedure are obtained. The iteration procedure yields a sequence of consistent estimates for the parameters of interest, $\beta^{(t)}$ and $\gamma^{(t)}=[\gamma_0^{(t)},\gamma_x^{(t)}]$ where $t=1,2,\ldots-1,T$ is the index for iteration and T denotes the iteration after which estimates converge. Denote the estimates after convergence as β and $\hat{\gamma}$. We are interested in obtaining the standard errors for $\hat{\beta}$ and $\hat{\gamma}$.

To obtain the standard errors for $\hat{\gamma}$, we derive the asymptotic distribution for $\hat{\gamma}$. $\hat{\gamma}$ is obtained from the following minimization problem (second part of the 3rd step of the proposed procedure in an iteration after convergence)

$$\min_{\gamma}\sum_{z=1}^{Z}-\log\left[\Pr\left(n_z^*|\gamma,\hat{\beta}\right)\right]=\min_{\gamma}\sum_{z=1}^{Z}\ell_z\left(\gamma,\hat{\beta}\right)$$

where ℓ_z is the shorthand for $-\log[\Pr(n_z^*|\gamma,\hat{\beta})]$ and Z is the number of auctions observed. Evaluated at the estimates, we have

$$\sum_{z=1}^{Z}s_z\left(\hat{\gamma},\hat{\beta}\right)=0$$

where $s_z=\frac{\partial\ell_z}{\partial\gamma}$. Denote β^* and γ^* as the true values for the parameters. Under standard regularity conditions, it can be shown (through mean value expansion around γ^*) (e.g. pp.353–356 of Wooldridge (2002)) that

$$\sqrt{Z}(\hat{\gamma}-\gamma^*)=\mathbf{A}_\gamma^{-1}\left[-Z^{-1/2}\sum_{z=1}^{Z}s_z\left(\gamma^*,\hat{\beta}\right)\right]+o_p(1)\tag{A.5}$$

where $\mathbf{A}_\gamma=E\left[\frac{\partial s_z\left(\gamma^*,\hat{\beta}\right)}{\partial\gamma'}\middle|\hat{\beta}=\beta^*\right]$. On the other hand, applying a mean value expansion to $Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\hat{\beta})$, we have

$$Z^{-1/2}\sum_{z=1}^{Z}s_z\left(\gamma^*,\hat{\beta}\right)=Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\beta^*)+\mathbf{F}_\gamma\sqrt{Z}\left(\hat{\beta}-\beta^*\right)+o_p(1)\tag{A.6}$$

where $F_\gamma=E\left[\frac{\partial s_z\left(\gamma^*\hat{\beta}\right)}{\partial\hat{\beta}'}\middle|\hat{\beta}=\beta^*\right]$. Plugging Eq. (A.6) into Eq. (A.5), we have

$$\sqrt{Z}\left(\hat{\gamma}-\gamma^*\right)=\mathbf{A}_\gamma^{-1}\left[-Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\beta^*)-\mathbf{F}_\gamma\sqrt{Z}\left(\hat{\beta}-\beta^*\right)\right]+o_p(1).\tag{A.7}$$

On the other hand, note that β is obtained from the following minimization problem (first part of the 3rd step of the proposed procedure in an iteration after convergence)

$$\min_{\beta}\sum_{b=1}^{B}-\log\left[\frac{f(v_b|\beta)}{1-F(p_0|\beta)}\right]=\min_{\beta}\sum_{b=1}^{B}\ell_b\left(\beta,\hat{\beta},\hat{\gamma}\right)$$

where v_b is calculated using $\hat{\beta}$ and $\hat{\gamma}$ as well as data and hence is a function of these variables. ℓ_b is the shorthand for $-\log\left[\frac{f(v_b|\beta)}{1-F(p_0|\beta)}\right]$. Evaluated at the estimates, we have

$$\sum_{b=1}^{B}s_b\left(\hat{\beta},\hat{\beta},\hat{\gamma}\right)=\sum_{b=1}^{B}s_b\left(\hat{\beta},\hat{\gamma}\right)=0$$

where $s_b\frac{\partial\ell_b}{\partial\beta}+\frac{\partial\ell_b}{\partial\beta}$. Again, under standard regularity conditions, mean value expansion around β^* yields

$$\sqrt{B}\left(\hat{\beta}-\beta^*\right)=\mathbf{A}_\beta^{-1}\left[-B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\hat{\gamma})\right]+o_p(1)\tag{A.8}$$

where $A_\beta=E\left[\frac{\partial s_b(\beta^*\hat{\gamma})}{\partial\beta'}\middle|\hat{\gamma}-\gamma^*\right]$. On the other hand, applying a mean value expansion to $B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\hat{\gamma})$, we have

$$B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\hat{\gamma})=B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\gamma^*)+\mathbf{F}_\beta\sqrt{B}(\hat{\gamma}-\gamma^*)+o_p(1)\tag{A.9}$$

where $\mathbf{F}_\beta=E\left[\frac{\partial s_b(\beta^*\hat{\gamma})}{\partial\hat{\gamma}'}\middle|\hat{\gamma}=\gamma^*\right]$. Plugging Eq. (A.9) into Eq. (A.8), we have

$$\sqrt{B}\left(\hat{\beta}-\beta^*\right)=\mathbf{A}_\beta^{-1}\left[-B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\gamma^*)-\mathbf{F}_\beta\sqrt{B}(\hat{\gamma}-\gamma^*)\right]+o_p(1).\tag{A.10}$$

Combining Eqs. (A.7) and (A.10) yields

$$\sqrt{Z}(\hat{\gamma}-\gamma^*)=\left(I_\gamma-\mathbf{A}_\gamma^{-1}\mathbf{F}_\gamma\mathbf{A}_\beta^{-1}\mathbf{F}_\beta\right)^{-1}\mathbf{A}_\gamma^{-1}\left[\begin{array}{c}-Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\beta^*)\\+\mathbf{F}_\gamma\mathbf{A}_\beta^{-1}\frac{\sqrt{Z}}{\sqrt{B}}B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\gamma^*)\end{array}\right]+o_p(1).$$

$$\sqrt{B}\left(\hat{\beta}-\beta^*\right)=\left(I_\beta-\mathbf{A}_\beta^{-1}\mathbf{F}_\beta\mathbf{A}_\gamma^{-1}\mathbf{F}_\gamma\right)^{-1}\mathbf{A}_\beta^{-1}\left[\begin{array}{c}-B^{-1/2}\sum_{b=1}^{B}s_b(\beta^*,\gamma^*)\\+\mathbf{F}_\beta\mathbf{A}_\gamma^{-1}\frac{\sqrt{B}}{\sqrt{Z}}Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\beta^*)\end{array}\right]+o_p(1).$$

Applying central limit theorem yields $Z^{-1/2}\sum_{z=1}^{Z}s_z(\gamma^*,\beta^*)\xrightarrow{d}Normal[0,E(s_zs_z')]$ and $B^{-1/2}\sum_{z=1}^{B}s_z(\beta^*,\gamma^*)\xrightarrow{d}Normal[0,E(s_bs_b')]$. As a result, the asymptotic distributions for $\sqrt{Z}(\hat{\gamma}-\gamma^*)$ and $\sqrt{B}\left(\hat{\beta}-\beta^*\right)$ are

$$\sqrt{Z}(\hat{\gamma}-\gamma^*)\xrightarrow{d}Normal\left\{0,C_\gamma\left[E(s_zs_z')+\mathbf{F}_\gamma\mathbf{A}_\beta^{-1}\frac{Z}{B}E(s_bs_b')\mathbf{A}_\beta^{-1}\mathbf{F}\mathbf{V}_\gamma\right]C_\gamma'\right\}\tag{A.11}$$

and

$$\sqrt{B}\left(\hat{\beta}-\beta^*\right)\xrightarrow{d}Normal\left\{0,C_\beta\left[E(s_bs_b')+\mathbf{F}_\beta\mathbf{A}_\gamma^{-1}\frac{Z}{B}E(s_zs_z')\mathbf{A}_\gamma^{-1}\mathbf{F}\mathbf{V}_\beta\right]C_\beta'\right\}\tag{A.12}$$

where $C_\gamma=(I_\gamma-\mathbf{A}_\gamma^{-1}\mathbf{F}_\gamma\mathbf{A}_\beta^{-1}\mathbf{F}_\beta)$ and $C_\beta=(I_\beta-\mathbf{A}_\beta^{-1}\mathbf{F}_\beta\mathbf{A}_\gamma^{-1}\mathbf{F}_\gamma)$, respectively. Finally, standard errors for $\hat{\gamma}$ and $\hat{\beta}$ are obtained using Eqs. (A.11) and (A.12) with γ^* and β^* replace by $\hat{\gamma}$ and $\hat{\beta}$ and expectations replaced by their sample counterparts.

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